

ISC Class 12 Mathematics Specimen Paper 1

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Maximum Marks: 80 | Time Allowed: Three hours | Reading Time: 15 minutes

Section A is compulsory. Attempt either Section B or Section C.

General Instructions

Write your answers on the separate answer paper provided. The first 15 minutes are for reading the paper, and you may not write during that time. Section A is compulsory and carries 65 marks. You must then attempt either Section B or Section C, each of which carries 15 marks, so that the paper you hand in carries $65 + 15 = 80$ marks in all. Do not attempt both Section B and Section C. Marks for each question, or part of a question, are shown in brackets. All working, including rough work, must be shown on the same sheet as the answer, and omission of essential working will lose marks. Mathematical tables and a list of formulae may be used. Graph paper is provided and must be used for the linear programming question if you attempt Section C. No diagrams or graphs are printed in this paper, so where a sketch is needed draw it yourself in pencil from the description given. This paper carries 80 of the 100 marks for the subject, the remaining 20 being for the project work.

SECTION A (65 Marks)

Attempt all questions from this Section.

Question 1 [15]

Each part carries 1 mark. For the parts that offer four choices, write only the option you choose. For the rest, give the answer in a single line.

- (i) The function f from the set of real numbers to itself given by $f(x) = 2x + 5$ is one to one and onto. Its inverse is:
(a) $(x + 5)/2$ (b) $2x - 5$ (c) $1/(2x + 5)$ (d) $(x - 5)/2$
- (ii) Write the principal value of $\sin^{-1}(\sin(3\pi/4))$.
- (iii) If A is a square matrix of order 3 and the determinant of A is 4, then the determinant of $2A$ is:
(a) 8 (b) 16 (c) 32 (d) 64
- (iv) A is a matrix of order 3×4 and B is a matrix of order 4×2 . State the order of the product AB .
- (v) The derivative of $\log(\sin x)$ with respect to x is:
(a) $\cot x$ (b) $\tan x$ (c) $\sec x$ (d) $\operatorname{cosec} x$
- (vi) The integral of $\sec^2(3x)$ with respect to x is:
(a) $3 \tan 3x + c$ (b) $\tan 3x + c$
(c) $(1/3) \tan 3x + c$ (d) $(1/3) \sec 3x + c$
- (vii) State the order and the degree of the differential equation $(d^2y/dx^2)^3 + (dy/dx)^5 + y = 0$.
- (viii) A and B are independent events with $P(A) = 0.3$ and $P(B) = 0.4$. Then $P(A \text{ and } B)$ is:
(a) 0.7 (b) 0.12 (c) 0.1 (d) 0.58
- (ix) For what value of x is the matrix whose first row is 2, 3 and whose second row is 4, x a singular matrix?
- (x) The function $f(x) = x^3$ on the set of real numbers is:
(a) increasing everywhere (b) decreasing everywhere
(c) increasing only for $x > 0$ (d) neither increasing nor decreasing anywhere
- (xi) Evaluate the definite integral of $\cos x$ with respect to x from 0 to $\pi/2$.
- (xii) The principal value branch of $\cos^{-1}x$ is:
(a) $[-\pi/2, \pi/2]$ (b) $[0, \pi]$

(c) $(0, \pi)$ (d) $[-\pi, \pi]$

(xiii) If $P(A \text{ and } B) = 1/6$ and $P(B) = 1/3$, write the value of $P(A \text{ given } B)$.

(xiv) The integrating factor of the differential equation $dy/dx + 2y = x$ is:

(a) e^x (b) e^{2x} (c) e^{-2x} (d) $2x$

(xv) A fair coin is tossed four times. The probability of getting exactly two heads is:

(a) $1/2$ (b) $1/4$ (c) $1/8$ (d) $3/8$

Question 2 [2]

Find the value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$, taking the principal value branch of $\cot^{-1}x$ to be the open interval from 0 to π . [2]

Question 3 [2]

A is the matrix whose first row is 3, 1 and whose second row is 5, 2. Find the determinant of A and hence find A^{-1} . [2]

Question 4 [2]

Differentiate x^x with respect to x, for $x > 0$. [2]

Question 5 [2]

Evaluate the definite integral of $x/(x^2 + 1)$ with respect to x from 0 to 1. [2]

Question 6 [2]

Solve the differential equation $dy/dx = (1 + y^2)/(1 + x^2)$. [2]

Question 7 [4]

Prove that $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$. [4]

Question 8 [4]

Without expanding along a row or a column at the start, show that the determinant whose first row is 1, a, a^2 , whose second row is 1, b, b^2 and whose third row is 1, c, c^2 is equal to $(a - b)(b - c)(c - a)$. [4]

Question 9 [4]

Find the intervals on which the function $f(x) = 2x^3 - 9x^2 + 12x + 5$ is increasing and the interval on which it is decreasing. [4]

Question 10 [4]

Evaluate the integral of $x^2 \log x$ with respect to x, for $x > 0$. [4]

Question 11 [6]

A is the matrix whose first row is 1, 1, 1, whose second row is 1, -1, 1 and whose third row is 2, 1, -1. Find the determinant of A, find A^{-1} by the adjoint method, and use A^{-1} to solve the system of equations $x + y + z = 6$, $x - y + z = 2$ and $2x + y - z = 1$. [6]

Question 12 [6]

An open topped box is to be made with a square base. The material available for the base and the four sides together measures 108 square centimetres, and the thickness of the material may be ignored. Taking the side of the base to be x centimetres, show that the volume of the box is $27x - x^3/4$ cubic centimetres, and find the dimensions of the box of greatest volume together with that greatest volume. Show by the second derivative test that your answer gives a maximum. [6]

Question 13 [6]

Solve the differential equation $dy/dx + 2y \tan x = \sin x$, given that $y = 0$ when $x = \pi/3$. [6]

Question 14 [6]

Three machines in a workshop turn out the same component. The first machine makes 25 per cent of the day's output, the second makes 35 per cent and the third makes the remaining 40 per cent. Of the components made by the first machine 5 per cent are defective, of those made by the second machine 4 per cent are defective and of those made by the third machine 2 per cent are defective. One component is drawn at random from the day's output and is found to be defective. Find the probability that it came from each of the three machines, and state which machine is the most likely source. [6]

SECTION B (15 Marks)

Attempt all questions from this Section if you have chosen Section B. Do not attempt Section C as well.

Question 15 [4]

The points A, B and C have position vectors $i + j + 2k$, $2i + 3j + 5k$ and $i + 5j + 5k$ respectively. Find the vector product of AB and AC, and hence find the area of the triangle ABC. [4]

Question 16 [5]

Find the shortest distance between the line through the point with position vector $i + 2j + k$ in the direction $i - j + k$ and the line through the point with position vector $2i - j - k$ in the direction $2i + j + 2k$. [5]

Question 17 [6]

Find the equation of the plane that passes through the three points $(2, 1, -1)$, $(-1, 3, 4)$ and $(1, -1, 2)$. Find also the perpendicular distance of this plane from the origin, and write down the direction ratios of the normal to the plane. [6]

SECTION C (15 Marks)

Attempt all questions from this Section if you have chosen Section C. Do not attempt Section B as well.

Question 18 [4]

- (a) The total cost in rupees of producing x units of an article is $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced. [2]
- (b) The total revenue in rupees from the sale of x units of a product is $R(x) = 60x - x^2$. Find the marginal revenue when 10 units are sold, and state what that figure means. [2]

Question 19 [5]

The following pairs of values were recorded for two variables x and y : $(2, 3)$, $(4, 7)$, $(6, 8)$, $(8, 10)$ and $(10, 12)$. Find the line of regression of y on x , and use it to estimate the value of y when x is 7. [5]

Question 20 [6]

A workshop makes two kinds of wooden stool, a plain stool and a cushioned stool. Each plain stool needs 2 hours of carpentry and 1 hour of finishing, while each cushioned stool needs 1 hour of carpentry and 3 hours of finishing. In a day the carpentry section can give at most 10 hours and the finishing section at most 15 hours. The profit is Rs 400 on a plain stool and Rs 300 on a cushioned stool. Taking x to be the number of plain stools and y the number of cushioned stools made in a day, write the problem as a linear programming problem. Use the graph paper provided to draw the feasible region for yourself, list its corner points, and find the number of each kind of stool that gives the greatest profit, stating that profit. [6]

ANSWER KEY AND MARKING SCHEME

Full working is given for every question. Marks shown in italics are the step marks an examiner would award.

Accept any correct alternative method that reaches the same answer. For the linear programming question accept a correctly drawn feasible region on any sensible scale, and accept corner points found either by reading the graph or by solving the boundary equations.

Question 1 [15]

- (i) (d) $(x - 5)/2$. Put $y = 2x + 5$, so $x = (y - 5)/2$, and hence $f^{-1}(x) = (x - 5)/2$.
- (ii) $\pi/4$. Since $3\pi/4$ lies outside the principal branch $[-\pi/2, \pi/2]$, write $\sin(3\pi/4) = \sin(\pi - 3\pi/4) = \sin(\pi/4)$, and $\pi/4$ does lie in the branch.
- (iii) (c) 32. For a square matrix of order n , the determinant of kA is k^n times the determinant of A , so here it is $2^3 \times 4 = 32$.
- (iv) The order of AB is 3×2 , the number of columns of A matching the number of rows of B .
- (v) (a) $\cot x$. The derivative is $(1/\sin x) \times \cos x = \cot x$.
- (vi) (c) $(1/3) \tan 3x + c$. Differentiating $(1/3) \tan 3x$ gives $(1/3) \times 3 \sec^2 3x = \sec^2 3x$.
- (vii) The order is 2, the highest derivative being the second, and the degree is 3, that being the power to which the second derivative is raised once the equation is free of radicals and fractions in the derivatives.
- (viii) (b) 0.12. For independent events $P(A \text{ and } B) = P(A) \times P(B) = 0.3 \times 0.4 = 0.12$.
- (ix) $x = 6$. The matrix is singular when its determinant is zero, so $2x - 12 = 0$ and $x = 6$.
- (x) (a) increasing everywhere. $f(x) = 3x^2$, which is never negative and is zero only at the single point $x = 0$, so the function rises throughout.
- (xi) The value is 1, since the integral of $\cos x$ is $\sin x$ and $\sin(\pi/2) - \sin 0 = 1 - 0 = 1$.
- (xii) (b) $[0, \pi]$.
- (xiii) $P(A \text{ given } B) = P(A \text{ and } B)/P(B) = (1/6)/(1/3) = 1/2$.
- (xiv) (b) e^{2x} . The equation is linear with $P = 2$, so the integrating factor is e to the power of the integral of 2 dx, that is e^{2x} .
- (xv) (d) $3/8$. The probability is ${}^4C_2 \times (1/2)^4 = 6/16 = 3/8$.

1 mark for each correct part. No mark is deducted for a wrong option, and a one-line answer carrying the correct mathematics gains the mark even if it is loosely worded.

Question 2 [2]

$\tan^{-1}(\sqrt{3}) = \pi/3$. For the second term, $\cot^{-1}(-\sqrt{3}) = \pi - \cot^{-1}(\sqrt{3}) = \pi - \pi/6 = 5\pi/6$, which lies in the branch $(0, \pi)$. Hence the value is $\pi/3 - 5\pi/6 = (2\pi - 5\pi)/6 = -\pi/2$.

$\cot^{-1}(-\sqrt{3}) = 5\pi/6$ 1. Final value $-\pi/2$ 1.

Question 3 [2]

The determinant of A is $3 \times 2 - 1 \times 5 = 6 - 5 = 1$. For a two by two matrix the inverse is got by swapping the diagonal entries, changing the sign of the other two and dividing by the determinant. So A^{-1} has first row 2, -1 and second row -5, 3. As a check, the product of A and A^{-1} has first row $3 \times 2 + 1 \times (-5)$, $3 \times (-1) + 1 \times 3$, that is 1, 0, and second row $5 \times 2 + 2 \times (-5)$, $5 \times (-1) + 2 \times 3$, that is 0, 1.

Determinant 1 1. Inverse with first row 2, -1 and second row -5, 3 1.

Question 4 [2]

Let $y = x^x$. Taking logarithms, $\log y = x \log x$. Differentiating both sides with respect to x , $(1/y)(dy/dx) = \log x + x \times (1/x) = 1 + \log x$. Hence $dy/dx = y(1 + \log x) = x^x(1 + \log x)$.

Taking logarithms and differentiating to get $(1/y)(dy/dx) = 1 + \log x$ 1. Answer $x^x(1 + \log x)$ 1.

Question 5 [2]

Put $u = x^2 + 1$, so $du = 2x dx$ and $x dx = du/2$. When $x = 0$, $u = 1$, and when $x = 1$, $u = 2$. The integral becomes one half of the integral of du/u from $u = 1$ to $u = 2$, that is $(1/2)[\log u]$ from 1 to 2 = $(1/2)(\log 2 - \log 1) = (1/2) \log 2$. The value is therefore $(\log 2)/2$, which is about 0.347.

Substitution $u = x^2 + 1$ with the limits changed 1. Value $(\log 2)/2$ 1.

Question 6 [2]

The equation separates as $dy/(1 + y^2) = dx/(1 + x^2)$. Integrating both sides, $\tan^{-1}y = \tan^{-1}x + c$, where c is an arbitrary constant. This may also be written as $y = \tan(\tan^{-1}x + c)$. Checking, if $y = \tan(\tan^{-1}x + c)$ then $dy/dx = \sec^2(\tan^{-1}x + c) \times 1/(1 + x^2) = (1 + y^2)/(1 + x^2)$, as required.

Variables separated and both sides integrated 1. Solution $\tan^{-1}y = \tan^{-1}x + c$ 1.

Question 7 [4]

Let $A = \tan^{-1}(1/2)$ and $B = \tan^{-1}(1/3)$, so that $\tan A = 1/2$ and $\tan B = 1/3$. Both A and B are positive and each is less than $\pi/4$, since $1/2$ and $1/3$ are both less than 1, so $A + B$ lies strictly between 0 and $\pi/2$. Using the addition formula, $\tan(A + B) = (\tan A + \tan B)/(1 - \tan A \tan B) = (1/2 + 1/3)/(1 - 1/6) = (5/6)/(5/6) = 1$. Since $A + B$ lies in the first quadrant and its tangent is 1, $A + B = \pi/4$. Hence $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$.

Setting $\tan A = 1/2$ and $\tan B = 1/3$ 1. Addition formula written down 1. Value of $\tan(A + B)$ found to be 1 1.

Conclusion with the quadrant justified 1.

Question 8 [4]

Subtract the second row from the first and the third row from the second. The determinant becomes the one whose first row is 0, $a - b$, $a^2 - b^2$, whose second row is 0, $b - c$, $b^2 - c^2$ and whose third row is 1, c , c^2 . Take $(a - b)$ out of the first row and $(b - c)$ out of the second, which leaves $(a - b)(b - c)$ multiplying the determinant whose first row is 0, 1, $a + b$, whose second row is 0, 1, $b + c$ and whose third row is 1, c , c^2 . Expanding this along the first column, only the entry 1 in the third row contributes, and its cofactor carries the sign $(-1)^{3+1} = +1$, giving the two by two determinant with first row 1, $a + b$ and second row 1, $b + c$. That equals $(b + c) - (a + b) = c - a$. Hence the whole determinant is $(a - b)(b - c)(c - a)$.

Two row operations carried out correctly 1. Factors $(a - b)$ and $(b - c)$ taken out 1. Expansion along the first column 1. Final factor $(c - a)$ and the stated result 1.

Question 9 [4]

$f(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$. The derivative vanishes at $x = 1$ and $x = 2$, and these two points cut the real line into three stretches. For $x < 1$, say $x = 0$, $f'(0) = 12$, which is positive. For $1 < x < 2$, say $x = 1.5$, $f'(1.5) = 6(0.5)(-0.5) = -1.5$, which is negative. For $x > 2$, say $x = 3$, $f'(3) = 6(2)(1) = 12$, which is positive. The function is therefore increasing on the interval from minus infinity to 1 and on the interval from 2 to infinity, and decreasing on the interval from 1 to 2.

Derivative factorised as $6(x - 1)(x - 2)$ 1. Critical points $x = 1$ and $x = 2$ 1. Sign of the derivative tested on all three stretches 1. Intervals of increase and decrease stated 1.

Question 10 [4]

Integrate by parts, taking $\log x$ as the first function and x^2 as the second. The integral equals $\log x \times (x^3/3)$ minus the integral of $(1/x)(x^3/3) dx$, that is $(x^3 \log x)/3$ minus the integral of $x^2/3 dx$, that is $(x^3 \log x)/3 - x^3/9 + c$. Checking by differentiation, the derivative of $(x^3 \log x)/3$ is $x^2 \log x + x^2/3$, and the derivative of $-x^3/9$ is $-x^2/3$, so the two extra terms cancel and $x^2 \log x$ is left.

Correct choice of first and second function 1. First stage of integration by parts 1. Second integral evaluated 1. Answer $(x^3 \log x)/3 - x^3/9 + c$ 1.

Question 11 [6]

Expanding along the first row, the determinant of A is $1 \times [(-1)(-1) - 1 \times 1] - 1 \times [1 \times (-1) - 1 \times 2] + 1 \times [1 \times 1 - (-1) \times 2] = 1 \times 0 - 1 \times (-3) + 1 \times 3 = 6$. Since this is not zero, A^{-1} exists. The cofactors are $A_{11} = 0$, $A_{12} = 3$, $A_{13} = 3$, $A_{21} = 2$, $A_{22} = -3$, $A_{23} = 1$, $A_{31} = 2$, $A_{32} = 0$ and $A_{33} = -2$. The adjoint is the transpose of the matrix of cofactors, so $\text{adj } A$ has first row 0, 2, 2, second row 3, -3, 0 and third row 3, 1, -2. Hence $A^{-1} = (1/6) \text{adj } A$, that is the matrix with first row 0, $1/3$, $1/3$, second row $1/2$, $-1/2$, 0 and third row $1/2$, $1/6$, $-1/3$. The system is $AX = B$ with B the column 6, 2, 1, so $X = A^{-1}B$. The first entry is $0 \times 6 + (1/3) \times 2 + (1/3) \times 1 = 1$, the second is $(1/2) \times 6 + (-1/2) \times 2 + 0 \times 1 = 2$, and the third is $(1/2) \times 6 + (1/6) \times 2 + (-1/3) \times 1 = 3 + 1/3 - 1/3 = 3$. So $x = 1$, $y = 2$ and $z = 3$. Checking in the original equations, $1 + 2 + 3 = 6$, $1 - 2 + 3 = 2$ and $2 + 2 - 3 = 1$, all of which hold.

Determinant 6 1. All nine cofactors 1. Adjoint as the transpose of the cofactor matrix 1. $A^{-1} = (1/6) \text{adj } A$ 1. Matrix product $X = A^{-1}B$ set up 1. Solution $x = 1, y = 2, z = 3$ 1.

Question 12 [6]

Let the side of the square base be x centimetres and the height h centimetres. The box is open at the top, so the material forms the base of area x^2 and four sides each of area xh . Hence $x^2 + 4xh = 108$, which gives $h = (108 - x^2)/4x$. The volume is $V = x^2h = x^2(108 - x^2)/4x = x(108 - x^2)/4 = 27x - x^3/4$, as required. Differentiating, $dV/dx = 27 - 3x^2/4$. Setting this to zero gives $3x^2 = 108$, so $x^2 = 36$ and $x = 6$, the negative root being rejected since x is a length. Then $d^2V/dx^2 = -3x/2$, which at $x = 6$ is -9 , a negative value, so $x = 6$ gives a maximum. The height is $h = (108 - 36)/24 = 72/24 = 3$ centimetres, and the greatest volume is $V = 27 \times 6 - 216/4 = 162 - 54 = 108$ cubic centimetres. The box therefore measures 6 centimetres by 6 centimetres by 3 centimetres.

Surface area equation $x^2 + 4xh = 108$ with h made the subject 1. Volume reduced to $27x - x^3/4$ 1. dV/dx found and set to zero 1. $x = 6$ obtained 1. Second derivative test showing a maximum 1. Height 3 cm and greatest volume 108 cm^3 1.

Question 13 [6]

The equation is linear in y with $P = 2 \tan x$ and $Q = \sin x$. The integral of $2 \tan x \, dx$ is $-2 \log|\cos x|$, so the integrating factor is e to that power, that is $|\cos x|^{-2}$, which we take as $\sec^2 x$. Multiplying through, $d/dx (y \sec^2 x) = \sin x \sec^2 x = \sin x / \cos^2 x$. Integrating the right hand side, put $u = \cos x$ so that $du = -\sin x \, dx$, and the integral becomes minus the integral of du/u^2 , that is $1/u$, that is $\sec x$. Hence $y \sec^2 x = \sec x + c$, so $y = \cos x + c \cos^2 x$. Applying the condition $y = 0$ at $x = \pi/3$, where $\cos x = 1/2$, gives $0 = 1/2 + c/4$, so $c = -2$. The solution is therefore $y = \cos x - 2 \cos^2 x$. Checking, $dy/dx = -\sin x + 4 \cos x \sin x$, and $2y \tan x = 2 \tan x \cos x - 4 \tan x \cos^2 x = 2 \sin x - 4 \sin x \cos x$. Adding the two gives $-\sin x + 4 \sin x \cos x + 2 \sin x - 4 \sin x \cos x = \sin x$, which is the right hand side.

Equation identified as linear with $P = 2 \tan x$ 1. Integral of $2 \tan x$ and the integrating factor $\sec^2 x$ 1. Equation written as $d/dx (y \sec^2 x) = \sin x \sec^2 x$ 1. Right hand side integrated to $\sec x$ 1. General solution $y = \cos x + c \cos^2 x$ 1. Constant $c = -2$ and the particular solution 1.

Question 14 [6]

Let E_1, E_2 and E_3 be the events that the component came from the first, the second and the third machine, and let D be the event that it is defective. Then $P(E_1) = 0.25$, $P(E_2) = 0.35$ and $P(E_3) = 0.40$, while $P(D \text{ given } E_1) = 0.05$, $P(D \text{ given } E_2) = 0.04$ and $P(D \text{ given } E_3) = 0.02$. By the law of total probability, $P(D) = 0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02 = 0.0125 + 0.0140 + 0.0080 = 0.0345$, which as a fraction is $69/2000$. By Bayes theorem, $P(E_1 \text{ given } D) = 0.0125/0.0345 = 125/345 = 25/69$, which is about 0.362. Likewise $P(E_2 \text{ given } D) = 0.0140/0.0345 = 28/69$, about 0.406, and $P(E_3 \text{ given } D) = 0.0080/0.0345 = 16/69$, about 0.232. These three add to $69/69 = 1$, as they must. The second machine is the most likely source, since $28/69$ is the largest of the three.

Prior probabilities and conditional probabilities set out 1. Law of total probability used 1. $P(D) = 0.0345$ 1. Bayes theorem stated or used correctly 1. All three posterior probabilities $25/69, 28/69$ and $16/69$ 1. Second machine named as the most likely source, with the three probabilities seen to add to 1 1.

Question 15 [4]

$AB = (2 - 1)i + (3 - 1)j + (5 - 2)k = i + 2j + 3k$, and $AC = (1 - 1)i + (5 - 1)j + (5 - 2)k = 0i + 4j + 3k$. The vector product is the determinant whose first row is i, j, k , whose second row is 1, 2, 3 and whose third row is 0, 4, 3. Expanding, the i component is $2 \times 3 - 3 \times 4 = 6 - 12 = -6$, the j component is $-(1 \times 3 - 3 \times 0) = -3$, and the k component is $1 \times 4 - 2 \times 0 = 4$. So $AB \times AC = -6i - 3j + 4k$. Its magnitude is $\sqrt{(36 + 9 + 16)} = \sqrt{61}$. The area of the triangle is half this, that is $(\sqrt{61})/2$ square units, which is about 3.91.

AB and AC found 1. Vector product set up as a determinant 1. $AB \times AC = -6i - 3j + 4k$ 1. Area $(\sqrt{61})/2$ 1.

Question 16 [5]

Write the lines as $r = a_1 + t d_1$ and $r = a_2 + s d_2$, with $a_1 = i + 2j + k$, $d_1 = i - j + k$, $a_2 = 2i - j - k$ and $d_2 = 2i + j + 2k$. The shortest distance between two skew lines is the modulus of $(a_2 - a_1)$ dotted with $(d_1 \times d_2)$, divided by the magnitude of $d_1 \times d_2$. Now $d_1 \times d_2$ is the determinant with first row i, j, k , second row 1, $-1, 1$ and third row 2, 1, 2. Its i component is $(-1) \times 2 - 1 \times 1 = -3$, its j component is $-(1 \times 2 - 1 \times 2) = 0$, and its k component is 1

$\times 1 - (-1) \times 2 = 3$. So $d_1 \times d_2 = -3i + 0j + 3k$, whose magnitude is $\sqrt{9 + 0 + 9} = 3\sqrt{2}$. Also $a_2 - a_1 = i - 3j - 2k$, and its dot product with $-3i + 0j + 3k$ is $-3 + 0 - 6 = -9$. Hence the shortest distance is $9/(3\sqrt{2}) = 3/\sqrt{2} = (3\sqrt{2})/2$, which is about 2.12 units. Since the numerator is not zero the lines are indeed skew.

Formula for the distance between skew lines quoted 1. $d_1 \times d_2 = -3i + 3k$ 1. Magnitude $3\sqrt{2}$ 1. Dot product with $a_2 - a_1$ equal to -9 1. Distance $3/\sqrt{2}$ 1.

Question 17 [6]

Call the points $P(2, 1, -1)$, $Q(-1, 3, 4)$ and $R(1, -1, 2)$. Then $PQ = -3i + 2j + 5k$ and $PR = -i - 2j + 3k$. A normal to the plane is $PQ \times PR$, the determinant with first row i, j, k , second row $-3, 2, 5$ and third row $-1, -2, 3$. Its i component is $2 \times 3 - 5 \times (-2) = 6 + 10 = 16$, its j component is $-[(-3) \times 3 - 5 \times (-1)] = -(-9 + 5) = 4$, and its k component is $(-3)(-2) - 2 \times (-1) = 6 + 2 = 8$. So the normal is $16i + 4j + 8k$, which is 4 times $4i + j + 2k$, and the direction ratios of the normal may be taken as 4, 1, 2. The plane through P with this normal is $4(x - 2) + 1(y - 1) + 2(z + 1) = 0$, that is $4x + y + 2z - 7 = 0$. Checking the other two points, at Q we get $-4 + 3 + 8 - 7 = 0$ and at R we get $4 - 1 + 4 - 7 = 0$, so all three points lie on it. The perpendicular distance from the origin is $|4 \times 0 + 0 + 2 \times 0 - 7|$ divided by $\sqrt{16 + 1 + 4} = 7/\sqrt{21}$, which may be written as $(\sqrt{21})/3$ and is about 1.53 units.

PQ and PR found 1. Vector product computed 1. Normal reduced to direction ratios 4, 1, 2 1. Equation $4x + y + 2z - 7 = 0$ 1. All three points verified to satisfy it 1. Distance from the origin $7/\sqrt{21}$ 1.

Question 18 [4]

- (a) The marginal cost is the derivative of the total cost. $C'(x) = 0.015x^2 - 0.04x + 30$. At $x = 3$, $C'(3) = 0.015 \times 9 - 0.04 \times 3 + 30 = 0.135 - 0.12 + 30 = 30.015$. The marginal cost when 3 units are produced is therefore Rs 30.015, or about Rs 30.02.

Derivative of the cost function 1. Value 30.015 at $x = 3$ 1.

- (b) The marginal revenue is the derivative of the total revenue. $R'(x) = 60 - 2x$, so $R'(10) = 60 - 20 = 40$. The marginal revenue when 10 units are sold is Rs 40, which is the approximate addition to total revenue brought in by selling the eleventh unit.

Derivative $60 - 2x$ with the value 40 1. Meaning of the figure stated 1.

Question 19 [5]

There are $n = 5$ pairs. The sum of the x values is $2 + 4 + 6 + 8 + 10 = 30$, so the mean of x is 6. The sum of the y values is $3 + 7 + 8 + 10 + 12 = 40$, so the mean of y is 8. The sum of the products xy is $6 + 28 + 48 + 80 + 120 = 282$, and the sum of x^2 is $4 + 16 + 36 + 64 + 100 = 220$. The regression coefficient of y on x is $b = (\text{sum of } xy - n \text{ times the product of the means}) / (\text{sum of } x^2 - n \text{ times the square of the mean of } x) = (282 - 5 \times 6 \times 8) / (220 - 5 \times 36) = (282 - 240) / (220 - 180) = 42/40 = 1.05$. The line of regression of y on x passes through the point of means, so $y - 8 = 1.05(x - 6)$, which gives $y = 1.05x + 1.7$. When $x = 7$, $y = 1.05 \times 7 + 1.7 = 7.35 + 1.7 = 9.05$.

Means 6 and 8 1. Sums 282 and 220 1. Regression coefficient 1.05 1. Line $y = 1.05x + 1.7$ 1. Estimate 9.05 at $x = 7$ 1.

Question 20 [6]

Let x be the number of plain stools and y the number of cushioned stools made in a day. The problem is to maximise $Z = 400x + 300y$ subject to the carpentry limit $2x + y \leq 10$, the finishing limit $x + 3y \leq 15$, and $x \geq 0$, $y \geq 0$. On graph paper draw the lines $2x + y = 10$, which joins $(5, 0)$ and $(0, 10)$, and $x + 3y = 15$, which joins $(15, 0)$ and $(0, 5)$. Shade the region that lies below both lines in the first quadrant. The two lines meet where $2x + y = 10$ and $x + 3y = 15$. From the first, $y = 10 - 2x$; substituting in the second gives $x + 30 - 6x = 15$, so $-5x = -15$ and $x = 3$, whence $y = 4$. The corner points of the feasible region are therefore $(0, 0)$, $(5, 0)$, $(3, 4)$ and $(0, 5)$. Evaluating Z at each, $Z(0, 0) = 0$, $Z(5, 0) = 2000$, $Z(3, 4) = 1200 + 1200 = 2400$ and $Z(0, 5) = 1500$. The greatest value is 2400 at the point $(3, 4)$, so the workshop should make 3 plain stools and 4 cushioned stools for a maximum profit of Rs 2400. Both resources are then used in full, since $2 \times 3 + 4 = 10$ hours of carpentry and $3 + 12 = 15$ hours of finishing.

Objective function and both constraints with the non negativity conditions 1. Both boundary lines plotted and the feasible region shaded 1. Point of intersection $(3, 4)$ found by solving the two equations 1. All four corner points listed 1. Z evaluated at every corner 1. Maximum Rs 2400 at 3 plain and 4 cushioned stools 1.