

ISC Class 12 Mathematics Specimen Paper 2

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Maximum Marks: 80 | Time Allowed: Three hours | Reading Time: 15 minutes

Section A is compulsory. Attempt either Section B or Section C.

General Instructions

Write your answers on the separate answer paper provided. The first 15 minutes are for reading the paper, and you may not write during that time. Section A is compulsory and carries 65 marks. You must then attempt either Section B or Section C, each of which carries 15 marks, so that the paper you hand in carries $65 + 15 = 80$ marks in all. Do not attempt both Section B and Section C. Marks for each question, or part of a question, are shown in brackets. All working, including rough work, must be shown on the same sheet as the answer, and omission of essential working will lose marks. Mathematical tables and a list of formulae may be used. Graph paper is provided and must be used for the linear programming question if you attempt Section C. No diagrams or graphs are printed in this paper, so where a sketch is needed draw it yourself in pencil from the description given. This paper carries 80 of the 100 marks for the subject, the remaining 20 being for the project work.

SECTION A (65 Marks)

Attempt all questions from this Section.

Question 1 [15]

Each part carries 1 mark. For the parts that offer four choices, write only the option you choose. For the rest, give the answer in a single line.

- (i) The functions f and g from the set of real numbers to itself are given by $f(x) = x + 1$ and $g(x) = x^2$. The value of $(g \text{ composed with } f)(2)$ is:
(a) 5 (b) 9 (c) 6 (d) 4
- (ii) Write the principal value of $\sin^{-1}(-1/2)$.
- (iii) If A is a square matrix of order 3 and the determinant of A is 5, then the determinant of the adjoint of A is:
(a) 5 (b) 15 (c) 125 (d) 25
- (iv) Write the inverse of the two by two matrix $2I$, where I is the unit matrix of order 2.
- (v) The derivative of $\tan^{-1}x$ with respect to x is:
(a) $1/(1 + x^2)$ (b) $1/(1 - x^2)$
(c) $1/(1 - x^2)$ (d) $-1/(1 + x^2)$
- (vi) Write the derivative of $\sin(x^2)$ with respect to x .
- (vii) The integral of $1/(x^2 + 9)$ with respect to x is:
(a) $3 \tan^{-1}(x/3) + c$ (b) $\tan^{-1}(3x) + c$
(c) $(1/9) \tan^{-1}(x/3) + c$ (d) $(1/3) \tan^{-1}(x/3) + c$
- (viii) Write the integral of e^{2x} with respect to x .
- (ix) The order and the degree of the differential equation $d^3y/dx^3 + 4(dy/dx)^2 + y = 0$ are:
(a) 3 and 2 (b) 2 and 3 (c) 1 and 3 (d) 3 and 1
- (x) Write the integrating factor of the differential equation $dy/dx + y/x = x^2$, for $x > 0$.
- (xi) For two events A and B , $P(A) = 0.6$, $P(B) = 0.5$ and $P(A \text{ or } B) = 0.8$. Then $P(A \text{ and } B)$ is:
(a) 0.1 (b) 0.2 (c) 0.3 (d) 0.4
- (xii) A and B are mutually exclusive events with $P(A) = 0.3$ and $P(B) = 0.45$. Write the value of $P(A \text{ or } B)$.

(xiii) The slope of the tangent to the curve $y = x^3 - 3x$ at the point where $x = 2$ is:

- (a) 2 (b) 5 (c) 9 (d) 12

(xiv) The function $f(x) = x^2 - 4x + 7$ has a minimum value at x equal to:

- (a) 0 (b) 2 (c) 4 (d) 7

(xv) Two fair dice are thrown together. Write the probability that the sum of the numbers shown is 9.

Question 2 [2]

Prove that $\sin^{-1}(3/5) + \sin^{-1}(4/5) = \pi/2$. [2]

Question 3 [2]

A is the matrix whose first row is 1, 2 and whose second row is 3, 4. Find the adjoint of A and verify that the product of A and its adjoint equals the determinant of A multiplied by the unit matrix of order 2. [2]

Question 4 [2]

Show that the derivative of $\log(\tan x)$ with respect to x is $2 \operatorname{cosec} 2x$. [2]

Question 5 [2]

Evaluate the integral of $x e^x$ with respect to x . [2]

Question 6 [2]

Solve the differential equation $dy/dx = e^{x+y}$. [2]

Question 7 [4]

A relation R is defined on the set of all integers by saying that a is related to b when $a - b$ is divisible by 5.

Show that R is an equivalence relation, and write down the equivalence class that contains 2. [4]

Question 8 [4]

Use matrices to solve the pair of equations $2x + 3y = 7$ and $3x - y = 5$. [4]

Question 9 [4]

Verify Lagrange's mean value theorem for the function $f(x) = x^2 - 3x + 2$ on the closed interval from 1 to 4, and find the value of c that the theorem provides. [4]

Question 10 [4]

Evaluate the integral of $e^x \sin x$ with respect to x . [4]

Question 11 [6]

Prove that the determinant whose first row is 1, x , x^3 , whose second row is 1, y , y^3 and whose third row is 1, z , z^3 is equal to $(x - y)(y - z)(z - x)(x + y + z)$. [6]

Question 12 [6]

A right circular cone has a fixed slant height l . Show that the cone of greatest volume is the one whose semi vertical angle has a tangent equal to $\sqrt{2}$, and find that greatest volume in terms of l . Use the second derivative test to confirm that your answer gives a maximum. [6]

Question 13 [6]

(a) Evaluate the integral of $x^2/[(x - 1)(x - 2)(x - 3)]$ with respect to x , using partial fractions. [4]

(b) Evaluate the definite integral of $1/(1 + x^2)$ with respect to x from 0 to 1. [2]

Question 14 [6]

Solve the differential equation $x^2 + y^2 = 2xy \, dy/dx$ by the substitution $y = vx$, and verify by differentiation that your solution satisfies the equation. [6]

SECTION B (15 Marks)

Attempt all questions from this Section if you have chosen Section B. Do not attempt Section C as well.

Question 15 [4]

The vectors a and b are given by $a = i + j - k$ and $b = 2i - j + 3k$. Find $a \times b$, find its magnitude, and hence write down a unit vector perpendicular to both a and b . [4]

Question 16 [5]

- (a) Find the angle between the line through the point $(1, -2, 3)$ in the direction $i + j + 2k$ and the plane $2x - y + z = 4$. [3]
- (b) Find the coordinates of the point at which that line meets that plane. [2]

Question 17 [6]

- (a) Find the scalar triple product of the vectors $2i - 3j + 4k$, $i + 2j - k$ and $3i - j + 2k$, and hence find the volume of the parallelepiped having them as three coterminal edges. [4]
- (b) Show that the vectors $i + 2j + 3k$, $2i + j + 2k$ and $3i + 3j + 5k$ are coplanar. [2]

SECTION C (15 Marks)

Attempt all questions from this Section if you have chosen Section C. Do not attempt Section B as well.

Question 18 [4]

- (a) The total cost in rupees of producing x units of a commodity is $C(x) = x^3/3 - 5x^2 + 30x + 15$. Find the marginal cost when 4 units are produced, and find the average cost when 3 units are produced. [2]
- (b) The demand function for a product is $p = 50 - 2x$, where p is the price in rupees and x the quantity demanded. Find the elasticity of demand when $x = 10$, and state whether the demand there is elastic or inelastic. [2]

Question 19 [5]

The following pairs of values were recorded for two variables x and y : $(1, 2)$, $(2, 5)$, $(3, 3)$, $(4, 8)$ and $(5, 7)$. Find both lines of regression, find the coefficient of correlation from the two regression coefficients, and estimate y when x is 6. [5]

Question 20 [6]

A dairy mixes two feeds, feed P and feed Q, for its cattle. Each kilogram of feed P supplies 1 unit of protein and each kilogram of feed Q supplies 2 units of protein, and the daily mix must supply at least 10 units of protein. Each kilogram of feed P occupies 3 units of storage space and each kilogram of feed Q occupies 4 units, and only 24 units of storage space are available. Feed P costs Rs 200 a kilogram and feed Q costs Rs 500 a kilogram. Taking x kilograms of feed P and y kilograms of feed Q, write the problem as a linear programming problem. Use the graph paper provided to draw the feasible region for yourself, list its corner points, and find the mix of least cost together with that cost. [6]

ANSWER KEY AND MARKING SCHEME

Full working is given for every question. Marks shown in italics are the step marks an examiner would award. Accept any correct alternative method that reaches the same answer. For the linear programming question accept a correctly drawn feasible region on any sensible scale, and accept corner points found either by reading the graph or by solving the boundary equations.

Question 1 [15]

- (i) (b) 9. (g composed with f)(2) = g(f(2)) = g(3) = $3^2 = 9$.
- (ii) $-\pi/6$, since $\sin(-\pi/6) = -1/2$ and $-\pi/6$ lies in the principal branch $[-\pi/2, \pi/2]$.
- (iii) (d) 25. For a square matrix of order n the determinant of the adjoint is the determinant of A raised to the power $n - 1$, so here it is $5^2 = 25$.
- (iv) The matrix 2I has 2 on both diagonal entries and zeros elsewhere, so its inverse is the matrix with $1/2$ on both diagonal entries and zeros elsewhere, that is $(1/2)I$.
- (v) (a) $1/(1 + x^2)$.
- (vi) By the chain rule the derivative is $\cos(x^2) \times 2x$, that is $2x \cos(x^2)$.
- (vii) (d) $(1/3) \tan^{-1}(x/3) + c$. Using the standard form with $a = 3$, the integral is $(1/a) \tan^{-1}(x/a) + c$.
- (viii) The integral is $e^{2x}/2 + c$, since differentiating $e^{2x}/2$ gives e^{2x} .
- (ix) (d) 3 and 1. The highest derivative is the third, so the order is 3, and that derivative occurs to the first power, so the degree is 1. The square on dy/dx does not affect the degree.
- (x) The equation is linear with $P = 1/x$. The integral of $1/x \, dx$ is $\log x$, so the integrating factor is $e^{\log x} = x$.
- (xi) (c) 0.3. From $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, $0.8 = 0.6 + 0.5 - P(A \text{ and } B)$, so $P(A \text{ and } B) = 1.1 - 0.8 = 0.3$.
- (xii) For mutually exclusive events the probabilities add, so $P(A \text{ or } B) = 0.3 + 0.45 = 0.75$.
- (xiii) (c) 9. $dy/dx = 3x^2 - 3$, and at $x = 2$ this is $12 - 3 = 9$.
- (xiv) (b) 2. $f'(x) = 2x - 4$, which is zero at $x = 2$, and $f''(x) = 2$, which is positive, so $x = 2$ gives a minimum. The minimum value itself is 3.
- (xv) There are 36 equally likely outcomes, and the sum is 9 for the four pairs (3, 6), (4, 5), (5, 4) and (6, 3). So the probability is $4/36 = 1/9$.

1 mark for each correct part. No mark is deducted for a wrong option, and a one-line answer carrying the correct mathematics gains the mark even if it is loosely worded.

Question 2 [2]

Let $A = \sin^{-1}(3/5)$, so $\sin A = 3/5$ and, since A lies in the first quadrant, $\cos A = 4/5$. Let $B = \sin^{-1}(4/5)$, so $\sin B = 4/5$ and $\cos B = 3/5$. Then $\sin(A + B) = \sin A \cos B + \cos A \sin B = (3/5)(3/5) + (4/5)(4/5) = 9/25 + 16/25 = 1$. Both A and B lie between 0 and $\pi/2$, so $A + B$ lies between 0 and π , and the only angle in that range whose sine is 1 is $\pi/2$. Hence $\sin^{-1}(3/5) + \sin^{-1}(4/5) = \pi/2$.

$\sin A$, $\cos A$, $\sin B$ and $\cos B$ found 1. $\sin(A + B) = 1$ with the conclusion $\pi/2$ 1.

Question 3 [2]

The determinant of A is $1 \times 4 - 2 \times 3 = 4 - 6 = -2$. The cofactors are $A_{11} = 4$, $A_{12} = -3$, $A_{21} = -2$ and $A_{22} = 1$, so the matrix of cofactors has first row 4, -3 and second row -2, 1, and its transpose, the adjoint, has first row 4, -2 and second row -3, 1. Multiplying, the product of A and adj A has first row $1 \times 4 + 2 \times (-3)$, $1 \times (-2) + 2 \times 1$, that is -2, 0, and second row $3 \times 4 + 4 \times (-3)$, $3 \times (-2) + 4 \times 1$, that is 0, -2. This is -2 times the unit matrix, that is the determinant of A times I, as required.

Adjoint with first row 4, -2 and second row -3, 1 1. Product shown equal to $-2I$ 1.

Question 4 [2]

Differentiating, $d/dx \log(\tan x) = (1/\tan x) \times \sec^2 x = \sec^2 x / \tan x$. Writing this in terms of sines and cosines, it is $(1/\cos^2 x) \times (\cos x / \sin x) = 1/(\sin x \cos x)$. Now $\sin 2x = 2 \sin x \cos x$, so $\sin x \cos x = (\sin 2x)/2$ and $1/(\sin x \cos x) = 2/\sin 2x = 2 \operatorname{cosec} 2x$. Hence the derivative is $2 \operatorname{cosec} 2x$.

Derivative obtained as $\sec^2 x / \tan x$ and reduced to $1/(\sin x \cos x)$ 1. Double angle formula used to reach $2 \csc 2x$ 1.

Question 5 [2]

Integrate by parts, taking x as the first function and e^x as the second. The integral equals $x e^x$ minus the integral of $1 \times e^x dx$, that is $x e^x - e^x + c$, which may be written $e^x(x - 1) + c$. Checking, the derivative of $e^x(x - 1)$ is $e^x(x - 1) + e^x = x e^x$.

Integration by parts applied with the right choice of functions 1. Answer $e^x(x - 1) + c$ 1.

Question 6 [2]

Write e^{x+y} as $e^x e^y$, so the equation separates as $e^{-y} dy = e^x dx$. Integrating both sides, $-e^{-y} = e^x + k$, which is usually written $e^x + e^{-y} = c$, where c is an arbitrary constant. Checking by differentiating the solution with respect to x , $e^x - e^{-y} dy/dx = 0$, so $dy/dx = e^x/e^{-y} = e^x e^y = e^{x+y}$, as required.

Variables separated using $e^{x+y} = e^x e^y$ 1. Solution $e^x + e^{-y} = c$ 1.

Question 7 [4]

Reflexive: for any integer a , $a - a = 0$, which is divisible by 5, so a is related to a . Symmetric: if a is related to b then $a - b = 5k$ for some integer k , so $b - a = -5k = 5(-k)$, which is again a multiple of 5, so b is related to a . Transitive: if a is related to b and b is related to c then $a - b = 5k$ and $b - c = 5m$ for integers k and m , so $a - c = (a - b) + (b - c) = 5(k + m)$, which is a multiple of 5, so a is related to c . Since R is reflexive, symmetric and transitive, it is an equivalence relation. The class containing 2 holds every integer b with $2 - b$ divisible by 5, that is every integer of the form $5n + 2$, so the class is the set $\{\dots, -8, -3, 2, 7, 12, \dots\}$.

Reflexive shown 1. Symmetric shown 1. Transitive shown 1. Equivalence class of 2 given as the integers of the form $5n + 2$ 1.

Question 8 [4]

Write the system as $AX = B$, where A has first row 2, 3 and second row 3, -1, X is the column x, y and B is the column 7, 5. The determinant of A is $2 \times (-1) - 3 \times 3 = -2 - 9 = -11$, which is not zero, so A^{-1} exists and the system has a unique solution. For a two by two matrix, $A^{-1} = (1/\det A)$ times the matrix with first row -1, -3 and second row -3, 2, which on dividing by -11 gives the matrix with first row $1/11, 3/11$ and second row $3/11, -2/11$. Then $X = A^{-1}B$ gives $x = (1/11) \times 7 + (3/11) \times 5 = (7 + 15)/11 = 22/11 = 2$ and $y = (3/11) \times 7 + (-2/11) \times 5 = (21 - 10)/11 = 11/11 = 1$. So $x = 2$ and $y = 1$. Checking, $2 \times 2 + 3 \times 1 = 7$ and $3 \times 2 - 1 = 5$, both of which hold.

System written as $AX = B$ 1. Determinant -11 with the note that the inverse exists 1. A^{-1} found correctly 1. Solution $x = 2, y = 1$ 1.

Question 9 [4]

The function $f(x) = x^2 - 3x + 2$ is a polynomial, so it is continuous on the closed interval from 1 to 4 and differentiable on the open interval from 1 to 4. Both conditions of the theorem are therefore met. Now $f(1) = 1 - 3 + 2 = 0$ and $f(4) = 16 - 12 + 2 = 6$, so the average rate of change is $[f(4) - f(1)]/(4 - 1) = (6 - 0)/3 = 2$. Also $f'(x) = 2x - 3$. Setting $2c - 3 = 2$ gives $2c = 5$ and $c = 5/2$, which lies strictly between 1 and 4. The theorem is thus verified, with $c = 5/2$.

Both conditions of the theorem checked 1. $f(1) = 0$ and $f(4) = 6$ with the average rate 2 1. Equation $2c - 3 = 2$ set up 1. $c = 5/2$ with the remark that it lies in the open interval 1.

Question 10 [4]

Let I be the required integral. Integrate by parts with e^x as the first function and $\sin x$ as the second. Then $I = -e^x \cos x +$ the integral of $e^x \cos x dx$. Integrate the new integral by parts in the same way, taking e^x first and $\cos x$ second, which gives the integral of $e^x \cos x dx = e^x \sin x - I$. Substituting back, $I = -e^x \cos x + e^x \sin x - I$, so $2I = e^x(\sin x - \cos x)$ and $I = e^x(\sin x - \cos x)/2 + c$. Checking by differentiation, the derivative of $e^x(\sin x - \cos x)/2$ is $[e^x(\sin x - \cos x) + e^x(\cos x + \sin x)]/2 = e^x(2 \sin x)/2 = e^x \sin x$.

First integration by parts 1. Second integration by parts 1. The original integral recovered and the equation in I formed 1. Answer $e^x(\sin x - \cos x)/2 + c$ 1.

Question 11 [6]

Subtract the second row from the first and the third row from the second. The first row becomes $0, x - y, x^3 - y^3$ and the second becomes $0, y - z, y^3 - z^3$, the third row being unchanged at $1, z, z^3$. Since $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ and similarly for the second row, take $(x - y)$ out of the first row and $(y - z)$ out of the second. This leaves $(x - y)(y - z)$ multiplying the determinant whose first row is $0, 1, x^2 + xy + y^2$, whose second row is $0, 1, y^2 + yz + z^2$ and whose third row is $1, z, z^3$. Expanding along the first column, only the 1 in the third row contributes, and its cofactor carries the sign $(-1)^{3+1} = +1$, giving the two by two determinant with first row $1, x^2 + xy + y^2$ and second row $1, y^2 + yz + z^2$. That equals $(y^2 + yz + z^2) - (x^2 + xy + y^2) = yz + z^2 - x^2 - xy = z(y + z) - x(x + y)$. Writing this as $(z - x)(x + y + z)$ by factorising, since $(z - x)(x + y + z) = zx + zy + z^2 - x^2 - xy - xz = zy + z^2 - x^2 - xy$, which agrees. Hence the determinant equals $(x - y)(y - z)(z - x)(x + y + z)$.

Two row operations carried out 1. Difference of cubes factorised in both rows 1. Factors $(x - y)$ and $(y - z)$ taken out 1. Expansion along the first column 1. Remaining expression simplified 1. Final factorisation $(z - x)(x + y + z)$ and the stated result 1.

Question 12 [6]

Let the radius of the base be r and the height h , with the slant height l fixed, so that $r^2 + h^2 = l^2$ and hence $r^2 = l^2 - h^2$. The volume is $V = (1/3)\pi r^2 h = (1/3)\pi(l^2 - h^2)h = (1/3)\pi(l^2 h - h^3)$. Differentiating with respect to h , $dV/dh = (1/3)\pi(l^2 - 3h^2)$. Setting this to zero gives $3h^2 = l^2$, so $h = l/\sqrt{3}$, the negative root being rejected. Differentiating again, $d^2V/dh^2 = (1/3)\pi(-6h) = -2\pi h$, which at $h = l/\sqrt{3}$ is negative, so the value gives a maximum. At that height $r^2 = l^2 - l^2/3 = 2l^2/3$, so $r = l\sqrt{2/3}$. If α is the semi vertical angle then $\tan \alpha = r/h = l\sqrt{2/3}$ divided by $l/\sqrt{3} = \sqrt{2/3} \times \sqrt{3} = \sqrt{2}$, which is the required result. The greatest volume is $V = (1/3)\pi(2l^2/3)(l/\sqrt{3}) = 2\pi l^3/(9\sqrt{3})$, which may also be written as $2\sqrt{3}\pi l^3/27$ and is about $0.403l^3$.

Relation $r^2 = l^2 - h^2$ used to express V in h alone 1. dV/dh found and set to zero 1. $h = l/\sqrt{3}$ obtained 1. Second derivative test showing a maximum 1. $\tan \alpha = \sqrt{2}$ shown 1. Greatest volume $2\pi l^3/(9\sqrt{3})$ 1.

Question 13 [6]

- (a) Write $x^2/[(x - 1)(x - 2)(x - 3)] = A/(x - 1) + B/(x - 2) + C/(x - 3)$. Multiplying through by the denominator, $x^2 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$. Putting $x = 1$ gives $1 = A(-1)(-2) = 2A$, so $A = 1/2$. Putting $x = 2$ gives $4 = B(1)(-1) = -B$, so $B = -4$. Putting $x = 3$ gives $9 = C(2)(1) = 2C$, so $C = 9/2$. Hence the integral is $(1/2)\log|x - 1| - 4\log|x - 2| + (9/2)\log|x - 3| + c$.

Partial fraction form set up 1. $A = 1/2$ 1. $B = -4$ and $C = 9/2$ 1. Integration giving the three logarithm terms 1.

- (b) The integral of $1/(1 + x^2)$ is $\tan^{-1}x$, so the definite integral is $\tan^{-1}1 - \tan^{-1}0 = \pi/4 - 0 = \pi/4$, which is about 0.785.

Antiderivative $\tan^{-1}x$ 1. Value $\pi/4$ 1.

Question 14 [6]

The equation may be written $dy/dx = (x^2 + y^2)/2xy$, in which the right hand side is homogeneous of degree zero, so put $y = vx$. Then $dy/dx = v + x dv/dx$, and the right hand side becomes $(x^2 + v^2x^2)/(2x \times vx) = (1 + v^2)/2v$. So $v + x dv/dx = (1 + v^2)/2v$, giving $x dv/dx = (1 + v^2)/2v - v = (1 + v^2 - 2v^2)/2v = (1 - v^2)/2v$. Separating, $2v dv/(1 - v^2) = dx/x$. The left hand side integrates to $-\log|1 - v^2|$ and the right to $\log|x|$, so $-\log|1 - v^2| = \log|x| + k$, that is $\log|x(1 - v^2)| = -k$, so $x(1 - v^2)$ is a constant. Putting $v = y/x$, $x(1 - y^2/x^2) = (x^2 - y^2)/x = C$, so the solution is $x^2 - y^2 = Cx$. To verify, differentiate $x^2 - y^2 = Cx$ with respect to x , giving $2x - 2y dy/dx = C$. Since $C = (x^2 - y^2)/x$, we have $2x - 2y dy/dx = (x^2 - y^2)/x$, so $2x^2 - 2xy dy/dx = x^2 - y^2$ and therefore $2xy dy/dx = 2x^2 - x^2 + y^2 = x^2 + y^2$, which is the original equation.

Equation recognised as homogeneous and $y = vx$ substituted 1. Reduced to $x dv/dx = (1 - v^2)/2v$ 1. Variables separated 1. Both sides integrated 1. Solution $x^2 - y^2 = Cx$ with v replaced by y/x 1. Verification by differentiation 1.

Question 15 [4]

$a \times b$ is the determinant whose first row is i, j, k , whose second row is $1, 1, -1$ and whose third row is $2, -1, 3$. The i component is $1 \times 3 - (-1)(-1) = 3 - 1 = 2$. The j component is $-[1 \times 3 - (-1) \times 2] = -(3 + 2) = -5$. The k component is $1 \times (-1) - 1 \times 2 = -1 - 2 = -3$. So $a \times b = 2i - 5j - 3k$, and its magnitude is $\sqrt{4 + 25 + 9} = \sqrt{38}$, which is about 6.16. A unit vector perpendicular to both a and b is therefore $(2i - 5j - 3k)/\sqrt{38}$, and the vector

with the opposite sign serves equally well. As a check, the dot product of $2\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$ with $\mathbf{a}$ is $2 - 5 + 3 = 0$, and with $\mathbf{b}$ is $4 + 5 - 9 = 0$.

Vector product set up as a determinant 1. $\mathbf{a} \times \mathbf{b} = 2\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$ 1. Magnitude $\sqrt{38}$ 1. Unit vector given 1.

Question 16 [5]

- (a) The direction vector of the line is $\mathbf{d} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and the normal to the plane is $\mathbf{n} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$. If θ is the angle between the line and the plane then $\sin \theta$ is the modulus of $\mathbf{d}$ dotted with $\mathbf{n}$, divided by the product of their magnitudes. Now $\mathbf{d}$ dotted with $\mathbf{n} = 2 - 1 + 2 = 3$, the magnitude of $\mathbf{d}$ is $\sqrt{1 + 1 + 4} = \sqrt{6}$, and the magnitude of $\mathbf{n}$ is $\sqrt{4 + 1 + 1} = \sqrt{6}$. So $\sin \theta = 3/(\sqrt{6} \times \sqrt{6}) = 3/6 = 1/2$, giving $\theta = 30$ degrees.

Formula with the sine of the angle and the normal quoted 1. Dot product 3 and both magnitudes $\sqrt{6}$ 1. Angle 30 degrees 1.

- (b) A general point on the line is $(1 + t, -2 + t, 3 + 2t)$. Substituting in $2x - y + z = 4$ gives $2(1 + t) - (-2 + t) + (3 + 2t) = 4$, that is $2 + 2t + 2 - t + 3 + 2t = 4$, so $3t + 7 = 4$ and $t = -1$. Putting $t = -1$ back gives the point $(0, -3, 1)$. Checking, $2 \times 0 - (-3) + 1 = 4$.

General point on the line substituted in the plane to give $t = -1$ 1. Point of intersection $(0, -3, 1)$ 1.

Question 17 [6]

- (a) The scalar triple product is the determinant whose first row is 2, -3, 4, whose second row is 1, 2, -1 and whose third row is 3, -1, 2. Expanding along the first row, it is $2[2 \times 2 - (-1)(-1)] - (-3)[1 \times 2 - (-1) \times 3] + 4[1 \times (-1) - 2 \times 3] = 2(4 - 1) + 3(2 + 3) + 4(-1 - 6) = 6 + 15 - 28 = -7$. The volume of the parallelepiped is the modulus of the triple product, that is 7 cubic units.

Triple product written as a determinant 1. Expansion carried out correctly 1. Value -7 1. Volume 7 cubic units 1.

- (b) Three vectors are coplanar when their scalar triple product is zero. The determinant here has first row 1, 2, 3, second row 2, 1, 2 and third row 3, 3, 5. Expanding along the first row, it is $1(1 \times 5 - 2 \times 3) - 2(2 \times 5 - 2 \times 3) + 3(2 \times 3 - 1 \times 3) = 1(5 - 6) - 2(10 - 6) + 3(6 - 3) = -1 - 8 + 9 = 0$. Since the triple product is zero the three vectors are coplanar. One may also note that the third vector is the sum of the first two, which shows the same thing.

Condition for coplanarity stated 1. Determinant shown to be zero 1.

Question 18 [4]

- (a) The marginal cost is $C'(x) = x^2 - 10x + 30$. At $x = 4$, $C'(4) = 16 - 40 + 30 = 6$, so the marginal cost is Rs 6. The average cost is $C(x)/x$. At $x = 3$, $C(3) = 27/3 - 45 + 90 + 15 = 9 - 45 + 90 + 15 = 69$, so the average cost is $69/3 = \text{Rs } 23$ per unit.

Marginal cost Rs 6 at $x = 4$ 1. Average cost Rs 23 at $x = 3$ 1.

- (b) From $p = 50 - 2x$ we get $dp/dx = -2$, so $dx/dp = -1/2$. The elasticity of demand is the modulus of $(p/x)(dx/dp)$. At $x = 10$, $p = 50 - 20 = 30$, so the elasticity is the modulus of $(30/10)(-1/2) = \text{the modulus of } -3/2 = 1.5$. Since this exceeds 1 the demand at that quantity is elastic.

Elasticity 1.5 1. Demand described as elastic because the value exceeds 1 1.

Question 19 [5]

There are $n = 5$ pairs. The sum of x is 15 and the sum of y is 25, so the mean of x is 3 and the mean of y is 5.

The sum of xy is $2 + 10 + 9 + 32 + 35 = 88$, the sum of x^2 is $1 + 4 + 9 + 16 + 25 = 55$, and the sum of y^2 is $4 + 25 + 9 + 64 + 49 = 151$. Now the sum of xy minus n times the product of the means is $88 - 5 \times 3 \times 5 = 88 - 75 = 13$. The sum of x^2 minus n times the square of the mean of x is $55 - 45 = 10$, and the sum of y^2 minus n times the square of the mean of y is $151 - 125 = 26$. So the regression coefficient of y on x is $13/10 = 1.3$ and the regression coefficient of x on y is $13/26 = 0.5$. The line of y on x passes through the means, so $y - 5 = 1.3(x - 3)$, which gives $y = 1.3x + 1.1$. The line of x on y is $x - 3 = 0.5(y - 5)$, which gives $x = 0.5y + 0.5$. The coefficient of correlation is the square root of the product of the two regression coefficients, taking the sign of the coefficients, so $r = \sqrt{(1.3 \times 0.5)} = \sqrt{0.65} = 0.806$, positive because both regression coefficients are positive. When $x = 6$, the line of y on x gives $y = 1.3 \times 6 + 1.1 = 7.8 + 1.1 = 8.9$.

Means 3 and 5 with the sums 88, 55 and 151 1. Regression coefficient of y on x equal to 1.3 1. Regression coefficient of x on y equal to 0.5 1. Both lines written out 1. Correlation coefficient 0.806 and the estimate 8.9 1.

Question 20 [6]

Let x be the kilograms of feed P and y the kilograms of feed Q. The problem is to minimise $Z = 200x + 500y$ subject to the protein requirement $x + 2y \geq 10$, the storage limit $3x + 4y \leq 24$, and $x \geq 0, y \geq 0$. On graph paper draw the line $x + 2y = 10$, which joins $(10, 0)$ and $(0, 5)$, and the line $3x + 4y = 24$, which joins $(8, 0)$ and $(0, 6)$. The feasible region is the set of points in the first quadrant lying on or above the first line and on or below the second. Note that it does not reach the x axis, since a point such as $(8, 0)$ satisfies the storage limit but gives only 8 units of protein, and $(10, 0)$ gives enough protein but needs 30 units of storage. The two lines meet where $x + 2y = 10$ and $3x + 4y = 24$. From the first, $x = 10 - 2y$; substituting in the second gives $30 - 6y + 4y = 24$, so $-2y = -6$ and $y = 3$, whence $x = 4$. The corner points are therefore $(0, 5)$, $(0, 6)$ and $(4, 3)$. Evaluating Z at each, $Z(0, 5) = 2500$, $Z(0, 6) = 3000$ and $Z(4, 3) = 800 + 1500 = 2300$. The least value is 2300 at the point $(4, 3)$, so the dairy should use 4 kilograms of feed P and 3 kilograms of feed Q at a least cost of Rs 2300. Both constraints are then tight, since $4 + 6 = 10$ units of protein and $12 + 12 = 24$ units of storage.

Objective function and both constraints with the non negativity conditions 1. Both boundary lines plotted and the feasible region shaded correctly 1. Point of intersection $(4, 3)$ found by solving the two equations 1. All three corner points listed 1. Z evaluated at every corner 1. Least cost Rs 2300 with 4 kg of P and 3 kg of Q 1.